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AVERAGED CHARACTERISTICS OF STRESSED LAMINATED MEDIA

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The article is concerned with the problem of calculating the average rigidity characteristics of an elastic laminated medium as functions of the initial stresses. The classical equations of an inhomogeneous elastic body with initial stresses are used as the starting ones, with the procedure of averaging (homogenization) applied to them. In the problem considered, the nonlinearity of the averaging procedure becomes very important and leads to a difference in the resulting working formulas from those derived in a classical manner.

It is noted in [1-3] that the averaging description of bodies with initial stresses requires the use of the method of averaging applied directly to the original inhomogeneous body. The use of a formula similar to that applied to homogeneous bodies (so-called "intermediate" averaging) generally gives an incorrect result. Works [1-3] were of a theoretical nature. In [3] it is noted that further progress in obtaining applied results can be made by considering specific structures.

The present work suggests an analysis of the problem of averaging for bodies of laminated structure. Such type of problems can be of interest for performing geophysical calculations, in particular, for taking into account the effect of initial stresses on the propagation of longwave vibrations. The results presented were partially reported in [4].

Statement of the Problem. Let us consider a laminated medium of periodic structure. The layers formed by homogeneous elastic materials are parallel to the Ox_1x_2 plane and have thickness $\varepsilon \ll 1$ (which is formalized as $\varepsilon \to 0$ [5]). We write the problem of elasticity theory for a body with initial stresses in the form [6]:

$$[(a_{ijkl}(x_3/\varepsilon) + \sigma^*_{jl}(\bar{x},\ x_3/\varepsilon)\,\delta_{ik})\, u^{\varepsilon}_{k,l}\ ]_{,j} = \rho\,(x_3/\varepsilon)\, u^{\varepsilon}_{i,tt} + f_i\,. \tag{1}$$

Here u^{ε} are the permutations; a_{ijkl} and ρ are the tensor of elastic constants and density, respectively; σ^*_{jl} is the tensor of initial stresses; f are the mass forces. The functions $a_{ijkl}(y_3)$, $\rho(y_3)$, $\sigma^*_{jl}(\bar{x}, y_3)$ are periodic in $y_3 = x_3/\varepsilon$ with period m (where m is the period of the structure of the considered body in dimensionless variables). We take the boundary conditions at the boundary of the region in the form:

$$\bar{u}^{\varepsilon}(\bar{x},\ t) = 0 \tag{2}$$

As shown below, the basic effects in averaging are not associated with boundary conditions. The initial conditions

$$\bar{u}^{\varepsilon}(\bar{x},\ 0) = \bar{u}^{\varepsilon}_{,t}(\bar{x},\ 0) = 0 \tag{3}$$

also do not influence the basic effects.

When $\varepsilon \to 0$, the inhomogeneous medium considered can be replaced by a certain homogeneous averaged medium [5] close in mechanical behavior to the original one. In the absence of initial stresses, the averaged body is described by the equations of elasticity theory with the so-called averaged elastic constants $A_{ijkl}(0)$ [5, 7]. In particular, these values are determined in experiments with macrospecimens of inhomogeneous media (i.e., specimens of size $\sim 1 \gg \varepsilon$).

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Let the initial stresses σ^*_{jl} be determined by solution of the problem

$$\sigma^*_{ij,j} = G_i \rho\,(x_3/\varepsilon)\,,\quad \sigma^*_{ij} = a_{ijkl}\,(x_3/\varepsilon)\, v^{\varepsilon}_{k,l} \tag{4}$$

with conditions (2) and (3). The averaging of problem (4), (2), and (3) has the form [5, 7]:

$$(A_{ijkl}\,(0)\, v_{k,l})_{,j} = G_i \langle \rho \rangle \tag{5}$$

with conditions (2) and (3). In this case, as shown, for example, in [7],

$$\langle \sigma^*_{ij} \rangle \to \sigma_{ij} = A_{ijkl}\,(0)\, v_{k,l}\,, \tag{6}$$

where $<\cdot> = m^{-1}\int_0^m dy_3$ is the period-mean of the structure; σ_{ij} are the averaged stresses. One of the possible suggestions for averaging problem (1) is the application of the formula (used in [6] for taking into account initial stresses) to a body with elastic constants $A_{ijkl}(0)$ and initial stresses σ_{jl} (so-called "intermediate" averaging). However, as follows from [1-3], in general this suggestion leads to an erroneous result and averaging of (1) leads to the equation

$$(A_{ijkl}\,(\sigma)\, u_{k,l})_{,j} = \langle \rho \rangle\, u_{,tt} + f_i\,, \tag{7}$$

where in general

$$A_{ijkl}\,(\sigma) \neq A_{ijkl}\,(0) + \sigma_{jl}\,\delta_{ik}\,. \tag{8}$$

We note that the formula used in [6] is applicable to the original (actual) inhomogeneous body. The inapplicability of the formula of [6] to an averaged body agrees with its fictitiousness (an averaged homogeneous body does not exist in reality).

In subsequent parts of the present work the statements formulated are justified at the level of the construction of a formal asymptotic expansion, working formulas are derived for stressed laminated media, and these formulas are examined.

Construction of an Averaged Problem. Introducing

$$\mathscr{A}_{ijkl}\,(\bar{x}\,,\ x_3/\varepsilon) = a_{ijkl}\,(x_3/\varepsilon) + \sigma^*_{jl}\,(\bar{x}\,,\ x_3/\varepsilon)\,\delta_{ik}\,, \tag{9}$$

it is possible to write (1) in a form similar to that of a system of equations from elasticity theory. Let us write (1) in variational form

$$\int_0^T \int_Q \mathscr{A}_{ijkl}\, u^{\varepsilon}_{k,l}\, \varphi_{i,j}\, d\bar{x}dt = \int_0^T \int_Q \rho\, \bar{u}^{\varepsilon}_{,tt}\, \bar{\varphi}\, d\bar{x}dt + \int_0^T \int_Q \bar{f}\, \bar{\varphi}\, d\bar{x}dt \tag{10}$$

for any $\bar{\varphi} \in \mathscr{D}\,([0,\, T],\, H^1(Q))$ (for the definition of spaces see [8]).

Let us introduce the two-scale expansion

$$\bar{u}^{\varepsilon} = \sum_{k=0}^{\infty} \varepsilon^k\, \bar{u}^{(k)}\,(\bar{x}\,,\ \bar{y})\,,\quad \bar{\varphi} = \sum_{k=0}^{\infty} \varepsilon^k\, \bar{\varphi}^{(k)}\,(\bar{x}\,,\ \bar{y})\,, \tag{11}$$

where $\bar{y} = \bar{x}/\varepsilon$ is a fast variable [7]. The functions of the variables $\bar{x}$, $\bar{y}$ are differentiated according to the rule [7]:

$$\frac{\partial}{\partial x_i} f\,(\bar{x}\,,\ \bar{y}) = \left(\frac{\partial}{\partial x_i} + \varepsilon^{-1}\, \frac{\partial}{\partial y_i} \right) f\,(\bar{x}\,,\ \bar{y})\,. \tag{12}$$

Substitution of (11), with account for (12), into (10) yields

$$\int_0^T\int_Q \mathscr{A}_{ijkl}\left(\frac{\partial u_k^{(0)}}{\partial x_l}+\frac{\partial u_k^{(1)}}{\partial y_l}+\ldots\right)\left(\frac{\partial \varphi_i^{(0)}}{\partial x_j}+\frac{\partial \varphi_i^{(1)}}{\partial y_j}+\ldots\right)d\bar{x}dt=$$

$$=\int_0^T\int_Q \rho\,(\bar{u}_{,tt}^{(0)}+\ldots)(\bar{\varphi}^{(0)}+\ldots)\,d\bar{x}dt+\int_0^T\int_Q \bar{f}(\bar{\varphi}^{(0)}+\ldots)\,d\bar{x}dt, \tag{13}$$

where the dots stand for terms of the order of ε and higher.

Let us take $\bar{\varphi}^{(0)}(\bar{x})=0$ and $\bar{\varphi}^{(1)}=\bar{\varphi}^{(1)}(\bar{y})\in\tilde{H}$, where $\tilde{H}$ denotes closure over the norm $H^1(P_1)$ of functions with a zero mean that are periodic in P_1. From this, we obtain, in turn, the problem of a periodicity cell:

$$\frac{\partial}{\partial y_j}\left[\mathscr{A}_{ijkl}(\bar{x},\bar{y})\left(\frac{\partial u_k^{(1)}}{\partial y_l}+\frac{\partial u_k^{(0)}(\bar{x})}{\partial x_l}\right)\right]=0, \tag{14}$$

$$\bar{u}^{(1)}\in\tilde{H},$$

and write the solution of (14) in the form

$$\bar{u}^{(1)}=\bar{N}^{(kl)}(\bar{x},\bar{y})\,\frac{\partial u_k^{(0)}(\bar{x})}{\partial x_l}, \tag{15}$$

where $\bar{N}^{kl}$ is the solution of (14) at $\bar{u}^{(0)}=x_k\bar{e}_l$. Substituting (15) into (13), at $\bar{\varphi}^{(1)}=0$ we obtain, as in [7],

$$\int_0^T\int_Q\left\langle \mathscr{A}_{ijkl}\left(\delta_p^k\delta_q^l+\frac{\partial N_p^{kl}}{\partial y_q}\right)\right\rangle\frac{\partial u_k^{(0)}}{\partial x_l}\,\frac{\partial \varphi_i^{(0)}(\bar{x})}{\partial x_j}\,d\bar{x}dt=$$

$$=\int_0^T\int_Q\langle\rho\rangle\,\bar{u}_{,tt}^{(0)}\bar{\varphi}^{(0)}(\bar{x})\,d\bar{x}dt+\int_0^T\int_Q\bar{f}\bar{\varphi}^{(0)}(\bar{x})\,d\bar{x}dt \tag{16}$$

for any $\bar{\varphi}^{(0)}\in\mathscr{D}([0,T],H^1(Q))$, i.e., we find the vibration equation for a body with the following averaged deformation characteristics:

$$A_{ijkl}(\sigma)=\left\langle\mathscr{A}_{ijkl}+\mathscr{A}_{ijpq}\frac{\partial N_p^{kl}}{\partial y_q}\right\rangle, \tag{17}$$

Here $\bar{N}^{kl}$ are determined from the solution of the problem

$$\frac{\partial}{\partial y_j}\left(\mathscr{A}_{ijkl}(\bar{x},\bar{y})\,\frac{\partial N_k^{pq}}{\partial y_l}+\mathscr{A}_{ijpq}(\bar{x},\bar{y})\right)=0,$$

$$\bar{N}^{pq}\in\tilde{H}. \tag{18}$$

In the absence of initial stresses, formulas (17) and (18) coincide with the formulas used in [5] for averaging bodies without initial stresses, while the coefficients $A_{ijkl}(0)$ are the averaged elastic constants of an unstressed body.

Laminated Media. If the layers are parallel to the $0x_1x_2$ plane, the functions a_{ijkl}, σ_{ij}^*, ρ depend on one fast variable y_3: $a_{ijkl}(y_3)$, $\sigma_{ij}^*(\bar{x},y_3)$, $\rho(y_3)$. In this case, Eq. (18) goes over into an ordinary differential equation:

$$(\mathcal{A}_{i3k3} N_k^{pq'} + \mathcal{A}_{i3pq})' = 0, \quad ' = \frac{\partial}{\partial y_3},$$

from which the following inequality is obtained:

$$N_k^{pq'} = -(\mathcal{A}_{i3k3})^{-1}(\mathcal{A}_{i3pq}) + (\mathcal{A}_{i3k3})^{-1}(C_i^{pq}),$$

where C_i^{pq} are constants determined from the periodicity condition for functions $\overline{N}^{pq}$ of the form $\langle\overline{N}^{pq'}\rangle = 0$. The reciprocal of the matrix $(\mathcal{A}_{i3k3})$ is denoted by $(\mathcal{A}_{i3k3})^{-1}$; $(\mathcal{A}_{i3pq})$ and (C_i^{pq}) are the vectors. We obtain

$$\langle(\mathcal{A}_{i3k3})^{-1}(\mathcal{A}_{i3pq})\rangle = \langle(\mathcal{A}_{i3k3})^{-1}\rangle(C_i^{pq}),$$

whence

$$C_i^{pq} = \langle(\mathcal{A}_{i3k3})^{-1}\rangle^{-1}\langle(\mathcal{A}_{k3m3})^{-1}(\mathcal{A}_{m3pq})\rangle$$

and

$$N_k^{pq'} = -(\mathcal{A}_{i3k3})^{-1}(\mathcal{A}_{i3pq}) + (\mathcal{A}_{i3k3})^{-1}\langle(\mathcal{A}_{i3l3})^{-1}\rangle^{-1}\langle(\mathcal{A}_{l3m3})^{-1}(\mathcal{A}_{m3pq})\rangle. \tag{19}$$

In the case considered, formula (17) takes the form:

$$A_{ijpq}(\sigma) = \langle\mathcal{A}_{ijpq} + \mathcal{A}_{ijk3} N_k^{pq'}\rangle. \tag{20}$$

Substitution of (19) into (20) gives

$$A_{ijpq}(\sigma) = \langle\mathcal{A}_{ijpq}\rangle - \langle(\mathcal{A}_{ijk3})(\mathcal{A}_{m3k3})^{-1}(\mathcal{A}_{m3pq})\rangle +$$

$$+\langle(\mathcal{A}_{ijk3})(\mathcal{A}_{n3k3})^{-1}\rangle\langle\mathcal{A}_{n3l3})^{-1}\rangle^{-1}\langle(\mathcal{A}_{l3m3})^{-1}(\mathcal{A}_{m3pq})\rangle. \tag{21}$$

Formula (21) is simplified, if media are considered whose layers are formed by orthotropic materials. For the latter, we write the matrix $(\mathcal{A}_{i3k3})$ as

$$\mathcal{A}_{i3k3} = a_{i3k3} + \overset{*}{\sigma}_{33}\delta_{ik} = 0 \quad \text{when} \quad i \neq k, \tag{22}$$

by virtue of which $(\mathcal{A}_{is3k3})^{-1} = (1/\mathcal{A}_{i3i3})$, and Eq. (21) takes the form

$$A_{ijpq}(\sigma) = \langle\mathcal{A}_{ijpq}\rangle - \left\langle\frac{\mathcal{A}_{ijk3}\ \mathcal{A}_{k3pq}}{\mathcal{A}_{k3k3}}\right\rangle + \frac{\left\langle\frac{\mathcal{A}_{ijk3}}{\mathcal{A}_{k3pq}}\right\rangle\left\langle\frac{\mathcal{A}_{k3pq}}{\mathcal{A}_{k3k3}}\right\rangle}{\left\langle\frac{1}{\mathcal{A}_{k3k3}}\right\rangle}. \tag{23}$$

We write expressions (22) for different values of the subscripts $ijkl$ with allowance for the definition of the quantities $\mathcal{A}_{ijkl}$:

$$A_{3333}(\sigma) = \frac{1}{\left\langle\frac{1}{a_{3333} + \overset{*}{\sigma}_{33}}\right\rangle}, \tag{24.1}$$

$$A_{3322}(\sigma) = \frac{\left\langle \dfrac{a_{3322}}{a_{3333} + \overset{*}{\sigma}_{33}} \right\rangle}{\left\langle \dfrac{1}{a_{3333} + \overset{*}{\sigma}_{33}} \right\rangle}, \tag{24.2}$$

$$A_{2323}(\sigma) = \frac{1}{\left\langle \dfrac{1}{a_{2323} + \overset{*}{\sigma}_{33}} \right\rangle}, \tag{24.3}$$

$$A_{3232}(\sigma) = \langle a_{3232} + \overset{*}{\sigma}_{22} \rangle - \left\langle \frac{a_{2323}^2}{a_{2323} + \overset{*}{\sigma}_{33}} \right\rangle + \frac{\left\langle \dfrac{a_{2323}}{a_{2323} + \overset{*}{\sigma}_{33}} \right\rangle^2}{\left\langle \dfrac{1}{a_{2323} + \overset{*}{\sigma}_{33}} \right\rangle}, \tag{24.4}$$

$$A_{1111}(\sigma) = \langle a_{1111} + \overset{*}{\sigma}_{11} \rangle - \left\langle \frac{a_{1133}^2}{a_{3333} + \overset{*}{\sigma}_{33}} \right\rangle + \frac{\left\langle \dfrac{a_{1133}}{a_{3333} + \overset{*}{\sigma}_{33}} \right\rangle^2}{\left\langle \dfrac{1}{a_{3333} + \overset{*}{\sigma}_{33}} \right\rangle}, \tag{24.5}$$

$$A_{1122}(\sigma) = \langle a_{1122} \rangle - \left\langle \frac{a_{1133}\, a_{2233}}{a_{3333} + \overset{*}{\sigma}_{33}} \right\rangle + \frac{\left\langle \dfrac{a_{3322}}{a_{3333} + \overset{*}{\sigma}_{33}} \right\rangle \left\langle \dfrac{a_{1133}}{a_{3333} + \overset{*}{\sigma}_{33}} \right\rangle}{\left\langle \dfrac{1}{a_{3333} + \overset{*}{\sigma}_{33}} \right\rangle}. \tag{24.6}$$

Concerning the Assignment of Initial Stresses. One way to reveal initial stresses $\overset{*}{\sigma}_{ij}$ is that of consideration of the vibrations of naturally loaded bodies (bodies under the force of gravity, etc.). Then the quantities $\overset{*}{\sigma}_{ij}$ are determined by solution of the problem of elasticity theory for bodies without initial stresses, i.e., by (1)-(3) at $\overset{*}{\sigma}_{ij} = 0$. For this case, the problem is well studied, and, in particular, it is known that for media of orthotropic materials

$$\langle \overset{*}{\sigma}_{ij} \rangle = \sigma_{ij}, \quad \overset{*}{\sigma}_{33} \text{ is independent of } y_3. \tag{25}$$

With account for (25), Eq. (24) yields the connection between the averaged coefficients $\mathcal{A}_{ijkl}(\sigma)$ and averaged initial stresses σ_{ij}. We give the corresponding formulas for the case of layers made of isotropic materials, expressing a_{ijkl} in terms of the Young modulus E and Poisson coefficient ν:

$$A_{3333}(\sigma) = \frac{1}{\left\langle \dfrac{(1+\nu)(1-2\nu)}{(1-\nu)E + \sigma_{33}(1+\nu)(1-2\nu)} \right\rangle}, \tag{26.1}$$

$$A_{3322}(\sigma) = A_{2233}(\sigma) = \frac{\left\langle \dfrac{E\nu}{(1-\nu)E + \sigma_{33}(1+\nu)(1-2\nu)} \right\rangle}{\left\langle \dfrac{(1+\nu)(1-2\nu)}{(1-\nu)E + \sigma_{33}(1+\nu)(1-2\nu)} \right\rangle}, \tag{26.2}$$

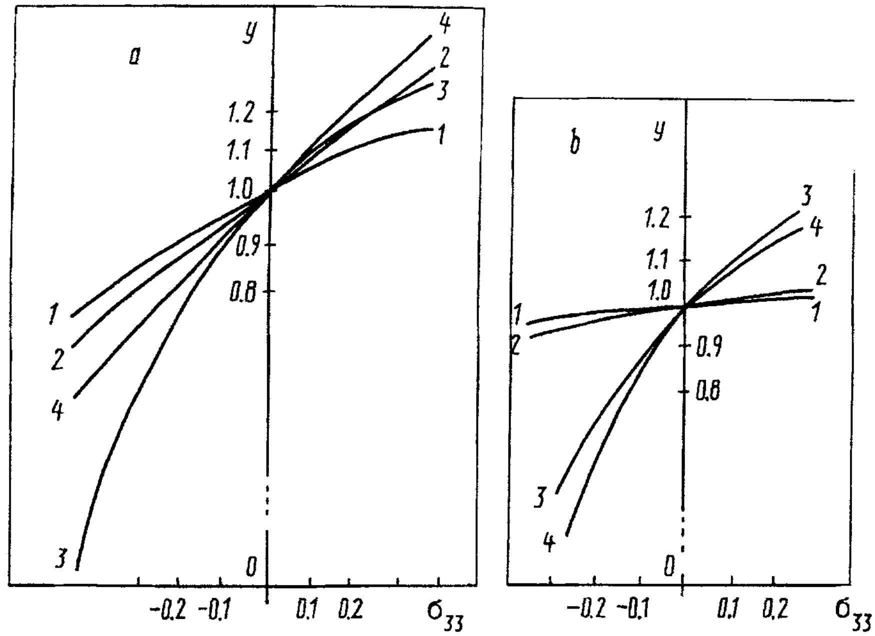

Fig. 1. A two-layer composite: a) $E_1 = 1 \cdot 10^{10}$; $\nu_1 = 0.1$; $\lambda_1 = 0.5$; $E_2 = 100 \cdot 10^{11}$; $\nu_2 = 0.3$; $\lambda_2 = 0.5$; values of subscripts in Eqs. (27): 1) 3333; 2) 3322; 3) 2323; 4) 3232; b) $E_1 = 1 \cdot 10^{11}$; $\nu_1 = 0.2$; $\lambda_1 = 0.25$; $E_2 = 10 \cdot 10^{10}$; $\nu_2 = 0.3$; $\lambda_2 = 0.75$; values of subscripts in Eq. (27): 1) 1111; 2) 1122; 3) 3131; 4) 1313.

$$A_{1313}(\sigma) = A_{2323}(\sigma) = \frac{1}{\left\langle \dfrac{1+\nu}{E + \sigma_{33}(1+\nu)} \right\rangle}, \tag{26.3}$$

$$A_{3232}(\sigma) = \left\langle \frac{E}{1+\nu} \right\rangle - \left\langle \frac{E^2}{(1+\nu)E + (1+\nu)^2 \sigma_{33}} \right\rangle + \frac{\left\langle \dfrac{E}{E + (1+\nu)\sigma_{33}} \right\rangle^2}{\left\langle \dfrac{1+\nu}{E + (1+\nu)\sigma_{33}} \right\rangle} + \sigma_{22}, \tag{26.4}$$

$$A_{1111}(\sigma) = \left\langle \frac{(1-\nu)E}{(1+\nu)(1-2\nu)} \right\rangle - \left\langle \frac{E^2\nu^2}{(1-\nu^2)(1-2\nu)E + (1+\nu)^2(1-2\nu)^2\sigma_{33}} \right\rangle +$$

$$+ \frac{\left\langle \dfrac{E\nu}{(1-\nu)E + (1+\nu)(1-2\nu)\sigma_{33}} \right\rangle^2}{\left\langle \dfrac{(1+\nu)(1-2\nu)}{E + (1-\nu)\sigma_{33}} \right\rangle} + \sigma_{11}, \tag{26.5}$$

$$A_{1122}(\sigma) = \left\langle \frac{E\nu}{(1+\nu)(1-2\nu)} \right\rangle - \left\langle \frac{E^2\nu^2}{(1-\nu^2)(1-2\nu)E + (1+\nu)^2(1-2\nu)^2\sigma_{33}} \right\rangle +$$

$$+ \frac{\left\langle \dfrac{E\nu}{(1-\nu)E + (1+\nu)(1-2\nu)\sigma_{33}} \right\rangle}{\left\langle \dfrac{(1+\nu)(1-2\nu)}{E + (1-\nu)\sigma_{33}} \right\rangle}. \tag{26.6}$$

As is seen, the dependences of $\mathcal{A}_{ijkl}(\sigma)$ on σ_{33} have a rather complex nonlinear character.

Figure 1 presents graphs of

$$y = A_{ijkl}(\sigma)/(A_{ijkl}(0) + \sigma_{jl}\delta_{ik}) \tag{27}$$

as a function of σ_{33} for $\sigma_{11} = \sigma_{22} = \sigma_{12} = 0$ for the values of the subscripts given in (26). The expression in the denominator of (27) corresponds to "intermediate averaging."

The Case of Small Initial Stresses. Very often the initial stresses appearing in a body are small compared with the characteristic value of the Young modulus of the layers E_l. In this case, in formulas (26) it is possible to carry out expansion in terms of the small parameter σ_{33}/E_l. Preserving only the linear terms, we obtain the following formulas:

$$A_{3333}(\sigma) = A_{3333}(0) + \sigma_{33} + \left[\frac{\left\langle \frac{(1+\nu)^2(1-2\nu)^2}{(1-\nu)^2 E^2} \right\rangle}{\left\langle \frac{(1+\nu)(1-2\nu)}{(1-\nu)E} \right\rangle^2} - 1\right]\sigma_{33}, \tag{28.1}$$

$$A_{3322}(\sigma) = A_{3322}(0) + \left[-\frac{\left\langle \frac{\nu(1+\nu)(1-2\nu)}{(1-\nu^2)E} \right\rangle}{\left\langle \frac{(1+\nu)(1-2\nu)}{(1-\nu)E} \right\rangle} + \left\langle \frac{\nu}{1-\nu} \right\rangle \frac{\left\langle \frac{(1+\nu)^2(1-2\nu)^2}{(1-\nu)^2 E^2} \right\rangle}{\left\langle \frac{(1+\nu)(1-2\nu)}{(1-\nu)E} \right\rangle^2}\right]\sigma_{33}, \tag{28.2}$$

$$A_{2323}(\sigma) = A_{2323}(0) + \sigma_{33} + \left[\frac{\left\langle \frac{(1+\nu)^2}{E^2} \right\rangle}{\left\langle \frac{1+\nu}{E} \right\rangle^2} - 1\right]\sigma_{33}, \tag{28.3}$$

$$A_{3232}(\sigma) = A_{3232}(0) + \sigma_{22} + \left[\frac{\left\langle \frac{(1+\nu)^2}{E^2} \right\rangle}{\left\langle \frac{1+\nu}{E} \right\rangle^2} - 1\right]\sigma_{33}, \tag{28.4}$$

$$A_{1111}(\sigma) = A_{1111}(0) + \sigma_{11} + \left[\left\langle \frac{\nu^2}{(1-\nu)^2} \right\rangle - 2\frac{\left\langle \frac{\nu(1+\nu)(1-2\nu)}{(1-\nu)^2 E} \right\rangle \left\langle \frac{\nu}{1-\nu} \right\rangle}{\left\langle \frac{(1+\nu)(1-2\nu)}{(1-\nu)E} \right\rangle} + \right.$$

$$\left. + \frac{\left\langle \frac{\nu}{1-\nu} \right\rangle^2 \left\langle \frac{(1+\nu)^2(1-2\nu)^2}{(1-\nu)^2 E^2} \right\rangle}{\left\langle \frac{(1+\nu)(1-2\nu)}{(1-\nu)E} \right\rangle^2}\right]\sigma_{33}, \tag{28.5}$$

$$A_{1122}(\sigma) = A_{1122}(0) + \left[\left\langle \frac{\nu^2}{(1-\nu)^2} \right\rangle - 2\frac{\left\langle \frac{\nu(1+\nu)(1-2\nu)}{(1-\nu)^2 E} \right\rangle \left\langle \frac{\nu}{1-\nu} \right\rangle}{\left\langle \frac{(1+\nu)(1-2\nu)}{(1-\nu)E} \right\rangle} + \right.$$

$$+\frac{\left\langle\frac{\nu}{1-\nu}\right\rangle^2\left\langle\frac{(1+\nu)^2(1-2\nu)^2}{(1-\nu)^2E^2}\right\rangle}{\left\langle\frac{(1+\nu)(1-2\nu)}{(1-\nu)E}\right\rangle^2}\Bigg]\sigma_{33}\,. \qquad (28.6)$$

The above formulas can be represented in the form of $A_{ijkl}(\sigma) = [A_{ijkl}(0) + \sigma_{jl}\delta_{ik}] + L_{ijklab}(E, \nu)\sigma_{ab}$. The term in the square brackets corresponds to "intermediate" averaging.

We give the formulas resulting from (28) for ν = const. They have a quite clear form and make it possible to evaluate the region of possible values of the coefficients L_{ijklab}. At ν = const, we write

$$A_{3333}(\sigma) = A_{3333}(0) + \sigma_{33} + L\sigma_{33}\,, \qquad (29)$$

$$A_{3322}(\sigma) = A_{3322}(0) + \frac{\nu}{1-\nu}L\sigma_{33}\,,$$

$$A_{2323}(\sigma) = A_{2323}(0) + \sigma_{33} + L\sigma_{33}\,,$$

$$A_{3232}(\sigma) = A_{3232}(0) + \sigma_{33} + L\sigma_{33}\,,$$

$$A_{1111}(\sigma) = A_{1111}(0) + \sigma_{11} + \frac{\nu^2}{(1-\nu)^2}L\sigma_{33}\,,$$

$$A_{2211}(\sigma) = A_{2211}(0) + \frac{\nu^2}{(1-\nu)^2}L\sigma_{33}\,,$$

where

$$L = \frac{\left\langle\frac{1}{E^2}\right\rangle}{\left\langle\frac{1}{E}\right\rangle^2} - 1\,. \qquad (30)$$

As is seen, the values of L_{ijklab} are proportional to L of (30). The question of the region of possible values for the quantities $<1/E^2>$ and $<1/E>$ is easily resolved by the methods of [9-11]. In particular, if no restrictions are imposed on E (except for those ensuring the existence of the indicated mean values), the quantity L (30) can take any positive values. A method for finding functions that provide functional (30) with a specified value is presented, in particular, in [9[. The mentioned method is also applicable for determining the possible values of the mean in (28).

NOTATION

a_{ijkl}, tensor of local elastic constants; E, ν, local Young modulus and Poisson coefficient; E_i, ν_i, λ_i, Young modulus, Poisson coefficient, and specific content of the i-th component in a composite material; σ^*_{il}, tensor of local initial stresses; $\sigma_{jl} = <\sigma^*_{jl}>$, period-mean value of the local initial stresses; $A_{ijkl}(\sigma)$, $A_{ijkl}(0)$, tensor of averaged elastic constants of a stressed and unstressed body; L_ε, differential operator of elasticity theory for an inhomogeneous body; $H^1_2(Q)$, $\mathcal{D}([0,T], H^1_2(Q))$, $\widetilde{H}$, functional spaces.

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